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On Three Notions of Orthosummability in Orthoalgebras

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We consider three notions of orthosummability for orthoalgebras that were recently introduced by Habil, Wilce, and Younce, respectively. Habil's orthosummability is shown to be equivalent to Wilce's. Younce's orthosummability is shown to be equivalent to Habil/Wilce's under the assumption that all blocks are suborthosummable.

1. INTRODUCTION

Noncommutative measure theory—the study of measures and states on algebraic structures less rich than a σ-field—arose from the realization that quantum mechanical events, instead of forming a σ-field, only form a σ-orthocomplete orthomodular lattice [Gu]. The nonexistence of a tensor product for such structures led to the inception of orthoalgebras, a less rich algebraic structure which admits a tensor product [Go].

In this section we will first introduce orthoalgebras (Definition 1.1) and show how they generalize more familiar ordered orthostructures (Lemmas 1.2 and 1.3). After this we introduce the three notions of orthosumability with which we are concerned (Definition 1.5).

Younce's notion is helpful when proving a result for the blocks first and then generalizing to a similar result for orthoalgebras. Wilce's notion is motivated by the connection between orthoalgebras and manuals. Habil's notion is a natural infinitary generalization of the summation $\oplus$. It also [Ha1, Lemma 3.8] extends the notion of orthocompleteness for orthomodular posets to the setting of orthoalgebras. We will use purely order-theoretic methods to show that Wilce's and Habil's notions are equivalent (Theorem 2.7) and

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that Younce's and Habil/Wilce's notions are "almost equivalent" (Theorem 2.5). Some of these results have for the countable case been established by Habil [Ha1, [Ha2] and Feldman and Wilce [FW]. Theorem 2.8 expresses Younce's notion of orthosummability in terms of chains in blocks. Younce's notion seems to be stronger than Habil/Wilce's as it also implies the suborthosummability of all blocks (Definition 1.6). However, there is still no example that shows Younce's notion to be genuinely stronger. To wit: There is currently no example of an orthoalgebra that is m-orthosummable in the sense of Habil/Wilce and that contains a block that is *not* Habil-sub-m-orthosummable. In Section 3 we give some examples which seem to indicate that Younce's notion should be genuinely stronger.

Definition 1.1 [Go, Ha1, Ha2]. An *orthoalgebra* is a quadruple $(L, \oplus, 0, 1)$, where L is a set containing two special elements 0, 1 and $\oplus$ is a partially defined binary operation on L that satisfies the following for all $p, q, r \in L$:

(OA1) (Commutativity). If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.
(OA2) (Associativity). If $q \oplus r$ and $p \oplus (q \oplus r)$ are defined, then $p \oplus q$ and $(p \oplus q) \oplus r$ are defined and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.
(OA3) (Orthocomplementation). For every $p \in L$ there is a unique $q \in L$ such that $p \oplus q$ is defined and $p \oplus q = 1$.
(OA4) (Consistency). If $p \oplus p$ is defined, then $p = 0$.

Lemma 1.2 [Ha1, Ha2]. Let L be an orthoalgebra.

(a) p, q are called *orthogonal* $(p \perp q)$ iff $p \oplus q$ is defined.
(b) L is a partially ordered set with the order defined as follows. For $p, q \in L$, we have $p \leq q$ iff there is an $r \perp p$ with $q = p \oplus r$.
(c) L is an orthoposet with the orthocomplement of p being the unique element $p' \in L$ with $p \oplus p' = 1$.
(d) L satisfies the *orthomodular identity*: If $p \leq q$, then $q = p \oplus (p \oplus q')'$. We set $q - p := (p \oplus q')'$.

Lemma 1.3 [Ha1, Ha2]:

(i) The orthoalgebra L is an orthomodular poset (OMP) if for any two $p \perp q$ the supremum $p \vee q$ exists.
(ii) The orthoalgebra L is an orthomodular lattice (OML) if for any two $p, q \in L$ the supremum $p \vee q$ exists.
(iii) The orthoalgebra L is a Boolean algebra if L is a distributive OML,
(iv) Any OMP, OML, or Boolean algebra is an orthoalgebra with $p \oplus q := p \vee q$ for any two p, q with $p \leq q'$.

Definition 1.4. Let L be an orthoalgebra. $M \subseteq L$ is a *suborthoalgebra* iff $0, 1 \in M$ and for all $p, q \in M$ we have that $p' \in M$ and if $p \oplus q$ is defined, $p \oplus q \in M$. Every suborthoalgebra is an orthoalgebra with the restrictions of the original operations. A suborthoalgebra that is a Boolean algebra is called a *Boolean subalgebra*. A maximal Boolean subalgebra is called a *block*. $X \subseteq L$ is called *jointly orthogonal* iff there is a block that contains X and any two elements of X are orthogonal. We let $J(L)$ be the set of all jointly orthogonal subsets of L.

Definition 1.5. Let L be an orthoalgebra and let m be a cardinal. Then L is called *m-orthosummable*:

(Ha) *in the sense of Habil* [Ha1, Ha2] iff for every $X \in J(L)$ with $|X| \leq m$ we have that

$$\oplus X := \bigvee_{F \in \mathcal{F}(X)}^{L} \oplus F$$

exists in L [$\mathcal{F}(X)$ denotes the set of all finite subsets of X].

(Wi) *In the sense of Wilce* [Wi] iff every chain $X \subseteq L$ with $|X| \leq m$ has a supremum in L.

(Yo) *In the sense of Younce* [Yo] iff for every $X \in J(L)$ with $|X| \leq m$ we have that for all blocks B of L with $X \subseteq B$, the supremum $\vee^{B} X$ exists and if A, B are two blocks with $X \subseteq A \cap B$, then $\vee^{A} X = \vee^{B} X$.

Definition 1.6 [Ha1, Definition 3.6]. Let L be an orthoalgebra that is m-orthosummable in the sense of Habil and let $A \subseteq L$ be a suborthoalgebra. Then A is called *Habil-sub-m-orthosummable* iff for each $X \subseteq A$ with $X \in J(L)$ and $|X| \leq m$ we have that $\oplus X \in A$.

2. RESULTS ON ORTHOSUMMABILITY

Lemma 2.1 (Cancellation law; [Ha1, Ha2]). Let L be an orthoalgebra. If $p, q \perp r$ and $p \oplus r \leq q \oplus r$, then $p \leq q$. ■

Lemma 2.2 ([Ha1, Lemma 3.14] or [Ha2, Lemma 4.11], easy modification). Let L be an orthoalgebra that is m-orthosummable in the sense of Habil. Assume each block of L is Habil-sub-m-orthosummable. Then L is m-orthosummable in the sense of Younce. ■

Lemma 2.3. Let L be an orthoalgebra and let $S \in J(L)$. Assume that for all blocks $A, B \supseteq S$ and all $T \subseteq S$ with $|T| < |S|$ the suprema $\vee^{B}T$ and $\vee^{A}T$ exist and are equal. Then there is a chain $C \subseteq L$ with the following properties:

1. For all blocks $B \subseteq L$ we have that $S \subseteq B$ iff $C \subseteq B$.
2. For all blocks $B \subseteq L$ that contain S and C

$$\{x \in B | \forall c \in C: x \geq c\} = \{x \in B | \forall s \in S: x \geq s\}$$

3. In every block $B \subseteq L$ with $B \supseteq C \cup S$ each $c \in C$ is the supremum of a subset $T \subseteq S$ with $|T| < |S|$.

4. For all $F \in \mathcal{F}(S)$ there is a $c \in C$ with $c \geq \bigoplus F$.

Proof. If S is finite, the claim is trivial. In case S is infinite, we argue as follows: Let B be a block that contains S. Let γ be the first ordinal with $|\gamma| = |S|$. Use γ to index S as $S = \{s_\alpha: \alpha < \gamma\}$. For each $\beta < \gamma$ we have $|\{s_\alpha: \alpha < \beta\}| = |\beta| < |\gamma| = |S|$. Hence we can define

$$c_\beta := \bigvee^{B} \{s_\alpha: \alpha < \beta\}$$

for all $\beta < \gamma$. Let

$$C := \{c_\beta: \beta < \gamma\}$$

Since the suprema that occur in the definition of the c_β are by hypothesis independent of the block in which they are taken, we infer that every block that contains S also contains C. Conversely, if A is a block that contains C, then A contains all $c_{\beta+1} - c_\beta$. However,

$$c_\beta \oplus (c_{\beta+1} - c_\beta) = c_{\beta+1} = c_\beta \overset{B}{\vee} s_\beta = c_\beta \oplus s_\beta$$

(all equalities are valid in B, hence in L) and hence by the cancellation law, Lemma 2.1, $s_\beta = c_{\beta+1} - c_\beta \in A$. This shows that $S \subseteq A$. Thus part 1 is proved.

Part 3 now follows from the hypothesis that the suprema were independent of the block in which they were taken. To prove 4 notice that for $F \in \mathcal{F}(S)$ we have $\bigoplus F = \vee^B F$. Find $\beta < \gamma$ such that $F \subseteq \{s_\alpha: \alpha < \beta\}$. Then $c_\beta \geq \bigoplus F$. Finally, it is easy to see that in any block $B \supseteq C \cup S$ every upper bound of C is an upper bound of S. Conversely, in every block $B \supseteq C \cup S$ each upper bound of S is bigger than all c_β and is hence an upper bound of C. This proves 2. ■

Lemma 2.4 ([Ha1, Corollary 3.2] or [Ha2, Lemma 4.1]). Let L be an orthoalgebra. Every chain $C \subseteq L$ is jointly compatible, i.e., there is a block $B \subseteq L$ with $C \subseteq B$.

Theorem 2.5. Let L be an orthoalgebra and let m be a cardinal. Then the following are equivalent:

1. L is m-orthosummable in the sense of Habil and each block of L is Habil-sub-m-orthosummable,
2. L is m-orthosummable in the sense of Younce.

Proof. By Lemma 2.2 we only have to prove that part 2 implies part 1. To do this we will prove by induction on $|X|$ that for every $X \in J(L)$ with

$|X| \leq m$ the generalized sum $\bigoplus X$ exists and for all blocks $B \supseteq X$ we have $\bigoplus X \in B$ and $\bigoplus X = \vee^B X$.

If X is finite, there is nothing to prove.

If X is infinite, assume that the statement is true for all $T \in J(L)$ with $|T| < |X|$. Let B be a block that contains X and let C be a chain in B as guaranteed by Lemma 2.3 for $S := X$. By Lemma 2.3, part 3, for each $c \in C$ there is a $T_c \subseteq X$ with $c = \vee^B T_c$ and $|T_c| < |X|$. Thus by induction hypothesis $c = \bigoplus T_c$. By part 2 of Lemma 2.3, $\vee^B X$ is an upper bound for C. Hence by part 4 of Lemma 2.3, $\vee^B X$ is an upper bound for all $\bigoplus F$, $F \in \mathcal{F}(X)$. Suppose p is another upper bound of all $\bigoplus F$, $F \in \mathcal{F}(X)$. Then $p \geq \bigoplus T_c = c$ for all $c \in C$. By Lemma 2.4 there is a block A such that $C \cup \{p\} \subseteq A$. By part 1 of Lemma 2.3, $X \cup C \cup \{p\} \subseteq A$. Thus $p \geq \vee^A X = \vee^B X$. This shows that $\vee^B X = \bigoplus X$. Since $\vee^B X$ is the supremum of X in any block that contains X we are done. ■

Lemma 2.6 ([Ha1, Theorem 3.5] or [Ha2, Theorem 4.4]). Let L be an orthoalgebra that is m-orthosummable in the sense of Habil. Then L is m-orthosummable in the sense of Wilce.

Theorem 2.7. Let L be an orthoalgebra and let m be a cardinal. The following are equivalent:

1. L is m-orthosummable in the sense of Habil.
2. L is m-orthosummable in the sense of Wilce.

Proof. By Lemma 2.6 we only need to prove that part 2 implies part 1. To do this we will prove the following: Let L be an orthoalgebra that is m-orthosummable in the sense of Wilce and let $X \in J(L)$ with $|X| \leq m$. Then $\bigoplus X$ exists and if X is infinite, it is equal to $\vee C$, where

$$C := \{\bigoplus\{x_\alpha: \alpha < \beta\}: \beta < \gamma\}$$

with $\{x_\alpha: \alpha < \gamma\}$ being an arbitrary indexing of X with γ, the first ordinal with $|\gamma| = |X|$.

Induction on $|X|$: If X is finite, there is nothing to prove.

Induction step: Let X be infinite, $|X| \leq m$, and let γ be the first ordinal with $|\gamma| = |X|$. The induction hypothesis is that for all $S \in J(L)$ with $|S| < |X|$, $\bigoplus S$ exists (and has a representation as above). Let $X = \{x_\alpha: \alpha < \gamma\}$ be an indexing as desired. By induction hypothesis the chain

$$C := \{\bigoplus\{x_\alpha: \alpha < \beta\}: \beta < \gamma\}$$

exists in L. Since L is m-orthosummable in the sense of Wilce and $|C| = |X| \leq m$, the supremum $s := \vee^L C$ exists. To see that s is an upper bound of $\{\bigoplus F: F \in \mathcal{F}(X)\}$, let $\{x_{\alpha_1}, \ldots, x_{\alpha_n}\} \in \mathcal{F}(X)$. Without loss of generality assume that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. Then $\{x_{\alpha_1}, \ldots, x_{\alpha_n}\} \subseteq \{x_\alpha: \alpha < \alpha_n + 1\}$ and hence

$$s \geq \bigoplus\{x_\alpha: \alpha < \alpha_n + 1\} \geq \bigoplus\{x_{\alpha_1}, \ldots, x_{\alpha_n}\}$$

Finally, to see that s is the desired supremum, let p be an upper bound of $\{\bigoplus F: F \in \mathcal{F}(X)\}$. Then for all $\beta < \gamma$, p is an upper bound of $\{\bigoplus F: F \in \mathcal{F}(\{x_\alpha: \alpha < \beta\})\}$. Hence $p \geq \bigoplus\{x_\alpha: \alpha < \beta\}$ for all $\beta < \gamma$, i.e., p is an upper bound of C, which implies $p \geq s$. ■

Theorem 2.8. Let L be an orthoalgebra and let m be a cardinal. The following are equivalent:

1. L is m-orthosummable in the sense of Habil and each block in L is Habil-sub-m-orthosummable.
2. Every chain $C \subseteq L$ with $|C| \leq m$ has a supremum $\vee^L C$ and every block that contains C also contains $\vee^L C$.

Proof. "1 ⇒ 2" is an easy modification of Corollary 4.10 in [Ha2]. To prove the converse, we will prove that part 2 implies L is m-orthosummable in the sense of Younce. Let $X \in J(L)$ with $|X| \leq m$. We need to prove that for all blocks $A, B \supseteq X$ the suprema $\vee^A X$ and $\vee^B X$ exist and are equal. We proceed by induction on $|X|$:

If X is finite, there is nothing to prove.

Induction step: Assume $X \in J(L)$ with $|X| \leq m$ and assume that the assertion holds for all $T \in J(L)$ with $|T| < |X|$. Let D be a block that contains X and apply Lemma 2.3 to $S := X$ to obtain a chain C. By assumption $\vee^L C$ is in every block $B \supseteq C$. Hence $\vee^B C = \vee^L C$ for all blocks $B \supseteq C$. By Lemma 2.3 every block B that contains X or C actually contains $X \cup C$ and X and C have the same upper bounds in B. Thus $\vee^B C = \vee^B X$. Finally if A, $B \supseteq X$ are blocks, then $\vee^A X = \vee^A C = \vee^L C = \vee^B C = \vee^B X$, and we are done. ■

3. EXAMPLES ON SUBORTHOSUMMABILITY

A question that remains open is whether in an orthoalgebra that is m-orthosummable in the sense of Habil every block is Habil-sub-m-orthosummable. A positive answer to this question would show that all three notions of orthosummability actually coincide. In [Ha1], Remark 3.7, an example is given that shows that not all Boolean subalgebras of an orthosummable orthoalgebra need be suborthosummable. No examples involving blocks were known. In the following we give an example of an orthoalgebra L and a block $B \subseteq L$ such that there is a chain $C \subseteq B$ that has a supremum in L, but no supremum in B (Example 3.2), and an example of an orthoalgebra L and a block $B \subseteq L$ such that there is a chain $C \subseteq B$ that has a supremum in L, a supremum in B, but the suprema do not coincide (Example 3.6). This shows that blocks need not be suborthosummable, as a supremum for a chain need not exist in the block and even if a chain has a supremum in a block, it need not be the supremum in L. However, the orthoalgebras in our examples are not orthosummable, hence this does not imply a negative answer to the above question.

For terminology regarding the paste job of two orthoalgebras we adhere to the terminology of the exposition in [Ha1], resp. [Ha3]:

Definition 3.1. Let L be an orthoalgebra. An *order ideal* is a nonempty subset $I \subseteq L$ such that for all $a, b \in L$ with $a \leq b$ we have that $b \in I$ implies $a \in I$. Let $I' := \{x \in L: x' \in I\}$. A *section* $S \subseteq L$ is a suborthoalgebra such that there is an order ideal I with $I \cap I' = \emptyset$ and $S = I \cup I'$.

Let L_1, L_2 be orthoalgebras and let $S_j := I_j \cap I_j' \subseteq L_j$ be sections. S_1, S_2 are called *corresponding sections* iff there exists an orthoalgebra-isomorphism θ: $S_1 \to S_2$ such that $\theta[I_1] = I_2$ and $\theta[I_1'] = I_2'$. Paste L_1 and L_2 together by identifying each $a \in S_1$ with $\theta(a) \in S_2$. For the equivalence classes define

$$[a] \oplus [b] := \begin{cases} [a_1 \oplus b_1] & \text{if} \quad [a] \cap L_1 = \{a_1\}, [b] \cap L_1 = \{b_1\}, \text{ and } a_1 \perp b_1 \\ [a_2 \oplus b_2] & \text{if} \quad [a] \cap L_2 = \{a_2\}, [b] \cap L_2 = \{b_2\}, \text{ and } a_2 \perp b_2 \end{cases}$$

As shown in [Ha1] and [Ha3] the object thus obtained is again an orthoalgebra, the "paste job" of L_1 and L_2 ([Ha3, Theorem 3.15]. Moreover if L_1 and L_2 are orthoposets, then the paste job is again an orthoposet [Ha3, Theorem 3.17].

Example 3.2. Let $\mathbf{R}$, $\mathbf{N}$ be the sets of real, resp. natural numbers. Let $\mathcal{F}(\mathbf{R})$, $\mathcal{CF}(\mathbf{R})$ denote the sets of finite, resp. cofinite subsets of R, and let $\mathcal{P}(X)$ denote the power set of X. Let

$$L_1 := (\mathcal{F}(\mathbf{R}) \cup \mathcal{CF}(\mathbf{R})) \times \{1\}$$

$$L_2 := (\mathcal{P}(\mathbf{N}) \cup \{\mathbf{R} \backslash S: S \in \mathcal{P}(\mathbf{N})\}) \times \{2\}$$

be two copies of Boolean subalgebras of $\mathcal{P}(\mathbf{R})$. If

$$I_j := (\mathcal{F}(\mathbf{R}) \cap \mathcal{P}(\mathbf{N})) \times \{j\} \subseteq L_j$$

then

$$I_j' = (\mathcal{CF}(\mathbf{R}) \cap \{\mathbf{R} \backslash S: S \in \mathcal{P}(\mathbf{N})\}) \times \{j\}$$

and the sections

$$S_j := I_j \cup I_j'$$

are corresponding sections. Let L be the orthoalgebra that is obtained by pasting L_1 and L_2 through the corresponding sections S_1 and S_2 (L actually is even an OMP). It can be proved that $L_1 \subseteq L$ actually is a block of L. However, the chain $C := \{\{1, \ldots, n\}: n \in \mathbf{N}\}$ does not have a supremum in L_1, while it clearly has $\mathbf{N}$ as a supremum in L.

The proofs of the next three lemmas are easy and will be left to the reader.

Lemma 3.3. Let P be an orthoposet and let $C \subseteq P$ be a chain that does not contain 1. Let $I_C := \{p \in L | \exists c \in C: c \geq p\}$ Then $S_C := I_C \cup I'_C$ is a section of L.

Lemma 3.4. Let L be an orthoalgebra (OMP, OML). Equip $L \times \{0, 1\}$ with the pointwise operations. Then $L \times \{0, 1\}$ is an orthoalgebra (OMP, OML).

Lemma 3.5. Let B be a Boolean algebra and let $a \in B$ be an atom. Then for all $b \in B$ we have $b \geq a$ or $b \leq a'$.

Example 3.6. Let $\mathcal{P}(\mathbf{N})$ be the power set of the natural numbers and let $T_n := \{1, \ldots, n\}$. Let $S := S_{\{T_n: n \in \mathbf{N}\}} = I_{\{T_n: n \in \mathbf{N}\}} \cup I'_{\{T_n: n \in \mathbf{N}\}}$ as in Lemma 3.3. Paste $\mathcal{P}(\mathbf{N})$ and $S \times \{0, 1\}$ together by identifying $I_{\{T_n: n \in \mathbf{N}\}}$ with $I_{\{T_n: n \in \mathbf{N}\}} \times \{0\}$ and $I'_{\{T_n: n \in \mathbf{N}\}}$ with $I'_{\{T_n: n \in \mathbf{N}\}} \times \{1\}$ and call the resulting orthoposet P. Then $(\mathbf{N}, 0) = \bigvee^P_{n \in \mathbf{N}} T_n$ and $(\mathbf{N}, 0)$ is a coatom in P. Let B be any block of P that contains $\mathcal{P}(\mathbf{N})$. Then by Lemma 3.5, B does not contain $(\emptyset, 1)$ (it would have to be an atom in B; however, it does not satisfy the property stated in Lemma 3.5) and hence $(\mathbf{N}, 0) \notin B$. Thus $\bigvee^B_{n \in \mathbf{N}} T_n = 1 \neq (\mathbf{N}, 0) = \bigvee^P_{n \in \mathbf{N}} T_n$.

Remark 3.7. We would have an example that Younce's σ-orthosummability is genuinely stronger than Habil/Wilce's if we could embed the above example into a σ-complete orthoalgebra in which $(\mathbf{N}, 0)$ still is a coatom and the supremum of the T_n.

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